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BOUNDS FOR ATTRACTORS AND THE EXISTENCE OF HOMOCLINIC ORBITS IN THE LORENZ SYSTEM†

G. A. LEONOV

St Petersburg

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Frequency estimates are derived for the Lyapunov dimension of attractors of non-linear dynamical systems. A theorem on the localization of global attractors is proved for the Lorenz system. This theorem is applied to obtain upper bounds for the Lyapunov dimension of attractors and to prove the existence of homoclinic orbits in the Lorenz system.

1. INTRODUCTION

The results obtained in [1] regarding the upper bounds for the Hausdorff dimension of attractors have stimulated the introduction of a new dimensional characteristic of invariant sets of dynamical systems – the Lyapunov dimension (LD) [2–7]. In the numerical analysis of many specific dynamical systems with chaotic behaviour, the LD, being an upper bound for the Hausdorff dimension, frequently turns out to be close to the Hausdorff dimension [2]. It has recently been ascertained that the fractal dimension also has an upper bound defined by the LD [8, 9]. Thus, having been defined in terms of the Lyapunov exponents, the concept of the LD is a connecting link between the classical theory of the stability of motion and the modern theory of dimensions for the attractors of dynamical systems.

In this paper frequency methods of analysing non-linear systems, developed primarily in the context of control theory [10–12], are used to find bounds for the LD of attractors. These bounds are used most effectively in combination with theorems stating that attractors are localized in a certain part of phase space. This approach is demonstrated here for the well-known Lorenz system, which is a three-mode model of two-dimensional convection [2, 13]. The localization bounds derived here for the global attractors of the Lorenz system are in many cases the asymptotically best possible. The dimension of these bounds with respect to the parameters has enabled asymptotic integration formulae to be used to prove the existence of homoclinic orbits.

We recall here a few definitions for the system

$$dx/dt = f(x), \quad x \in \mathbf{R}^n \tag{1.1}$$

where $f(x)$ is a smooth vector function and $\mathbf{R}^n$ is Euclidean n-space.

Definition 1. We shall say that a set $K \subset \mathbf{R}^n$ is *invariant* if, for any point $x_0 \in K$, it is true that $x(t, x_0) \in K, \forall t \in (-\infty, +\infty)$.

Definition 2. We shall say that a set $K \subset \mathbf{R}^n$ is *globally attracting* if, for any solution $x(t, x_0)$ of system (1.1)

$$\lim_{t \to +\infty} \inf_{u \in K} |x(t, x_0) - u| = 0$$

where $|\cdot|$ is the Euclidean norm in $\mathbf{R}^n$.

Definition 3. We shall say that the set K is *uniformly globally attracting* if, for any sphere $B \subset \mathbf{R}^n$ and any number $\varepsilon > 0$, a number $\tau(B, \varepsilon) > 0$ exists such that $x(t, x_0) \in K_\varepsilon\ \forall t \geqslant \tau(B, \varepsilon), \forall x_0 \in B$, where K_ε is the ε-neighbourhood of the set K.

Definition 4. A globally attracting bounded invariant set K is said to be a *global attractor* of system (1.1).

Definition 5. A uniformly globally attracting bounded invariant set K is said to be a *global B-attractor* of system (1.1).

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Let $F_t z$ denote the operator of displacement along trajectories of system (1.1)

$$F_t z = x(t, z), \quad x(0, z) = z$$

It is well known that the matrix

$$X(t, z) = \partial F_t / \partial z, \quad X(0, z) = I$$

is a fundamental matrix of the linear system

$$dy / dt = \partial f / \partial x \,|_{x=x(t,z)}\, y \tag{1.2}$$

where $\delta f/\delta x$ is the Jacobian of the vector function $f(x)$ at the point x.

Let $\alpha_1(t, z) \geqslant \ldots \geqslant \alpha_n(t, z)$ denote the singular numbers of the matrix $X(t, z)$. We recall that a singular number $\alpha_j(t, z)$ of the matrix $X(t, z)$ is defined as the square root of the corresponding eigenvalue $\rho_j(t, z)$ of the matrix $X(t, z)^* X(t, z)$, where $\rho_1(t, z) \geqslant \ldots \geqslant \rho_n(t, z)$ and the asterisk denotes transposition in the real case and Hermitian conjugation in the complex case.

Let $\omega_j(t, z)$ denote the product of the singular numbers

$$\omega_j(t, z) = \alpha_1(t, z) \ldots \alpha_j(t, z)$$

We successively introduce the following notation

$$\mu_1(z) = \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln \omega_1(t, z), \quad \mu_j(z) = \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln \omega_j(t, z) - M_{j-1}(z)$$
$$M_j(z) = \mu_1(z) + \ldots + \mu_j(z)$$

it is obvious that

$$\mu_j(z) \leqslant \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln \alpha_j(t, z) \leqslant \overline{\lim_{t\to+\infty}} \frac{1}{t} \ln \omega_j(t, z)^{1/j} \leqslant \frac{1}{j} M_j(z) \tag{1.3}$$

Let $\sup_{z\in K} \mu_1(z) \geqslant 0$ and let $d \in [1, n-1]$ be the least integer such that

$$M_{d+1}(z) < 0, \quad \forall z \in K \tag{1.4}$$

Henceforth, we will consider sets K such that inequality (1.4) holds for some $d \in [1, n-1]$. In that case it follows from inequalities (1.3) that the following definition is correct.

Definition 6. The number

$$\dim_L K = d + \sup_{z\in K}(M_d(z) / |\mu_{d+1}(z)|)$$

is known as the *Lyapunov dimension* $\dim_L K$ of the set K.

2. FREQUENCY ESTIMATES OF THE LYAPUNOV DIMENSION OF INVARIANT SETS

To estimate $\dim_L K$ for invariant sets of non-linear systems (1.1), we need estimates of the singular numbers $\alpha_1(t) \geqslant \ldots \geqslant \alpha_n(t)$ of the fundamental matrix $Y(t)$ ($Y(0) = I$) of the linear system

$$dy/dt = A(t)y, \quad y \in \mathbf{R}^n \tag{2.1}$$

where $A(t)$ is a continuous $n \times n$ matrix.

Let $u_1(t), \ldots, u_n(t)$ be an orthonormal system of eigenvectors of the matrix $Y(t)^* Y(t)$ such that

$$Y(t)^* Y(t) u_j(t) = \alpha_j(t)^2 u_j(t)$$

We recall that the linear operator $Y(t)$ maps the orthonormal system $u_1(t), \ldots, u_n(t)$ into an orthogonal system and $|Y(t)u(t)| = \alpha_j(t)$

Let us consider some linear k-dimensional subspace $L_k \subset \mathbf{R}^n$ and a non-negative function $\varphi(t)$.

Lemma 1. Suppose, for some vector $v \in L_k$, we have an estimate

$$|Y(t)v| \geqslant \varphi(t)|v|$$

Then $\alpha_k(t) \geqslant \varphi(t)$.

Proof. Fix the parameter t and choose a non-zero vector $v \in L_k$ such that $v^* u_j(t) = 0\ \forall_j = 1, \ldots, k-1$. It follows that v admits of the following expansion in terms of the basis $u_1(t), \ldots, u_n(t)$

$$v = \sum_{j=k}^{n} \beta_j u_j(t), \quad \beta_j \in \mathbf{R}^1$$

Using this equality, the orthonormality of $u_j(t)$ and the assumption of the lemma, we obtain

$$\varphi(t)^2 |v|^2 \leqslant |Y(t)v|^2 = \sum_{j=k}^{n} \beta_j^2 \alpha_j(t)^2 \leqslant \alpha_k(t)^2 |v|^2$$

Let H be a symmetric matrix which has at least k negative eigenvalues; consider the quadratic form $V(x) = x^* Hx$ and the set $\Omega = \{x \mid x^* Hx < 0\}$.

Lemma 2. Let us assume that for some function $\lambda(t)$ and any vector $z \in \Omega$

$$z^* HA(t)z + \lambda(t)z^* Hz \leqslant 0, \quad \forall t \geqslant 0 \tag{2.2}$$

Then a positive number β exists such that

$$\alpha_k(t) \geqslant \beta E(t), \quad \forall t \geqslant 0; \quad E(t) = \exp\left(-\int_0^t \lambda(\tau)d\tau\right) \tag{2.3}$$

Proof. We write inequality (2.2) in the form

$$(V(y(t))E^{-2}(t))^{\cdot} \leqslant 0$$

Hence it follows that

$$V(y(t)) \leqslant V(y(0))E^2(t)$$

From this inequality and the bounds on the spectrum of the matrix H we deduce the existence of a k-dimensional linear subspace L_k and a number β such that

$$|y(t)| \geqslant \beta|y(0)|E(t), \quad \forall y(0) \in L_k$$

By Lemma 1, this estimate implies inequality (2.3).

Now let M be a symmetric matrix which has at least $\varkappa$ positive eigenvalues; consider the quadratic form $W(x) = x^* Mx$ and the set $\Phi = \{x \mid x^* Mx > 0\}$.

Lemma 3. Let us assume that for some function $\psi(t)$ and any vector $z \in \Phi$,

$$z^* A(t)z + \psi(t)z^* Mz \leqslant 0, \quad \forall t \geqslant 0 \tag{2.4}$$

Then a positive number C exists such that

$$\alpha_{n-k+1}(t) \leqslant C\exp\left(-\int_0^t \psi(\tau)d\tau\right), \quad \forall t \geqslant 0 \tag{2.5}$$

Proof. Consider the system

$$\dot{z} = -A(t)^* z$$

and the fundamental matrix $Z(t)$ of the system satisfying the initial condition $Z(0) = I$. It is well known that [14]

$$Z(t)^* Z(t) = (Y(t)^* Y(t))^{-1}$$

Consequently, $\gamma_j(t) = \alpha_{n-j+1}(t)^{-1}$, where $\gamma_j(t)$ are the singular numbers of the matrix $Z(t)$, so that $\gamma_1(t) \geqslant \ldots \geqslant \gamma_n(t)$.
We introduce the notation

$$H = -M, \quad \lambda(t) = -\psi(t), \quad A_0(t) = -A(t)^*, \quad C = \beta^{-1}$$

Then condition (2.4) takes the form (2.2) with the matrix $A(0) = A_0(t)$.
Using Lemma 2, we obtain inequality (2.5).

Let us consider system (2.1) with a matrix $A(t) = A + BU(t)$, where A is a constant $n \times n$ matrix, B is a constant $n \times m$ matrix and $U(t)$ is a continuous $m \times n$ matrix.
Now consider a Hermitian form $F_k(z, \xi)$ of complex vector variables $z \in C^n$, $\xi \in C^m$ $(k = 1, \ldots, n)$
Let us assume that an $n \times m$ matrix Q_k and a positive number ε exist such that

$$F_k(x, Q_k^* x) \geqslant \varepsilon \, | Q_k^* x |^2, \quad \forall x \in \mathbf{R}^n \tag{2.6}$$

We also assume that

$$F_k(x, U(t)x) \geqslant 0, \quad \forall x \in \mathbf{R}^n, \quad \forall t \geqslant 0 \tag{2.7}$$

Lemma 4. Suppose the pair (A, B) is completely controllable, the pairs (A, Q_k) are completely observable and, for some sequence of numbers $\lambda_1 \leqslant \ldots \leqslant \lambda_n$, the following conditions are satisfied:
1) The matrices $A + BQ_k^* + \lambda_k I$ have at least k eigenvalues with positive real parts.
2) For all $k = 1, \ldots, n$ and all $\omega \in \mathbf{R}^1$

$$F_k([(i\omega - \lambda_k)I - A]^{-1} B\xi, \xi) \leqslant 0, \quad \forall \xi \in C^m \tag{2.8}$$

Then if $\alpha_k(t)$ are the singular numbers of the fundamental matrix of system (2.1), numbers $\beta_k > 0$ exist such that

$$\alpha_k(t) \geqslant \beta_k \exp(-\lambda_k t), \quad \forall t \geqslant 0, k = 1, \ldots, n \tag{2.9}$$

We recall that a pair (A, B) is said to be completely controllable if the rank of the matrix $(B, AB, \ldots, A^{n-1}B)$ is n. A pair (A, Q) is said to be completely observable if the pair (A^*, Q) is completely controllable.

Proof. It follows from Condition 2 of the lemma that, by the Yakubovich-Kalman Lemma, a symmetric matrix H_k exists for which the following inequality holds

$$2z^* H_k[(A + \lambda_k I)z + B\xi] + F_k(z, \xi) \leqslant 0, \quad \forall z \in \mathbf{R}^n, \quad \forall \xi \in \mathbf{R}^m \tag{2.10}$$

Putting $\xi = Q_k^* z$ in this inequality and using inequality (2.6), we obtain

$$2z^* H_k(A + \lambda_k I + BQ_k^*)z \leqslant -\varepsilon \, | Q_k^* z |^2, \quad \forall z \in \mathbf{R}^n$$

This inequality and Condition 1 of the lemma imply that the matrix H_k has at least k negative eigenvalues [11, 12].
It follows from condition (2.7) and inequality (2.10) that the function $V_k(y) = y^* H_k y$ and any solution $y(t)$ of system (1.2) will satisfy the relations

$$\begin{aligned} &\dot{V}_k(y(t)) + 2\lambda_k V_k(y(t)) = 2y(t)^* H_k[(A + \lambda_k I)y(t) + BU(t)y(t)] + \\ &+ F_k(y(t), \; U(t)y(t)) - F_k(y(t), \; U(t)y(t)) \leqslant 0, \quad \forall t \geqslant 0 \end{aligned} \tag{2.11}$$

By Lemma 2, this implies the estimate (2.9).

Lemma 5. Let Condition 1 in the statement of Lemma 4 be replaced by the following: the matrices $A + BQ_k + \lambda_k I$ have k or more eigenvalues with negative real parts.

Then, if $\alpha_k(t)$ are the singular numbers of the fundamental matrix of system (2.1), numbers $\beta_k > 0$ exist such that

$$\alpha_{n-k+1}(t) \leqslant \beta_k \exp(-\lambda_k t), \quad \forall t \geqslant 0, \quad k = 1, \dots, n \tag{2.12}$$

The proof is analogous to that of Lemma 4, except that Lemma 3 is used instead of Lemma 2.

Now consider the system

$$dx/dt = Ax + Bg(x), \quad x \in \mathbf{R}^n \tag{2.13}$$

where A and B are constant $n \times n$ and $n \times m$ matrices, and $g(x)$ is a continuously differentiable vector function.

Let K be a bounded set, invariant with respect to system (2.13), whose elements satisfy the estimate:

$$\gamma(x) + \dot{v}(x) \leqslant -\gamma_0, \quad \forall x \in K; \quad \gamma(x) = \mathrm{tr}(A + B\partial g/\partial x) \tag{2.14}$$

where $\delta g/\delta x$ is the Jacobian of the vector function $g(x)$ at the point x, γ_0 is a certain positive number and $v(x)$ is some function, continuously differentiable in $\mathbf{R}^n$, such that

$$\dot{v}(x) = (Ax + Bg(x))^* \operatorname{grad} v(x)$$

We will also assume here that inequalities (2.6) hold for certain Hermitian forms $F_k(z, \xi)$ and matrices Q_k. Instead of inequalities (2.7), we will assume here that

$$F_k(y, (\partial g/\partial x)y) \geqslant 0, \quad \forall y \in \mathbf{R}^n, \quad \forall x \in K \tag{2.15}$$

Theorem 1. Let the pair (A, B) be completely controllable and the pairs (A, Q_k) completely observable and suppose Conditions 1 and 2 of Lemma 4 hold for some sequence of positive numbers $\lambda_1 \leqslant \dots \leqslant \lambda_n$.

Suppose for some natural number m and number $s \in [0, 1]$

$$\begin{aligned} &-\gamma_0 + (1-s)\lambda_{m+1} + \Lambda_{m+2}^n < 0 \quad \text{for} \quad m < n-1; \quad \Lambda_m^n = \sum_{j=m}^{n} \lambda_j \\ &-\gamma_0 + (1-s)\lambda_{m+1} < 0 \quad \text{for} \quad m = n-1 \end{aligned} \tag{2.16}$$

Then

$$\dim_L K \leqslant m + s \tag{2.17}$$

Proof. Consider the case $m < n - 1$. We will first show that

$$M_{m+1}(z) < 0, \quad \forall z \in K \tag{2.18}$$

To do this, we use the relations

$$\omega_n(t, z) = \exp\int_0^t \gamma(x(\tau, z))d\tau = \left[\exp\int_0^t (\gamma(x(\tau, z)) + \dot{v}(x(\tau, z)))d\tau\right] \frac{\exp v(z)}{\exp v(x(t, z))} \leqslant$$

$$\leqslant C\exp(-\gamma_0 t), \quad C = \sup_{x \in K} \exp v(x) / \inf_{z \in K} \exp v(z)$$

$$\omega_{m+1}(t, z) = \omega_n(t, z)\alpha_{m+2}(t, z)^{-1} \dots \alpha_n(t, z)^{-1}$$

Hence it follows, by Lemma 4, that

$$\omega_{m+1}(t, z) \leqslant C \prod_{j=m+1}^{n} \beta_j^{-1} \exp[(-\gamma_0 + \Lambda_{m+2}^n)t], \quad \forall t \geqslant 0 \tag{2.19}$$

Taking into account that the numbers λ_j are positive, we deduce from this inequality and from condition (2.16) that a positive number ε exists for which

$$\omega_{m+1}(t, z) \leqslant C \prod_{j=m+2}^{n} \beta_j^{-1} \exp(-\varepsilon t) \tag{2.20}$$

Since

$$M_{m+1}(z)=\overline{\lim_{t\to+\infty}}\frac{1}{t}\ln\omega_{m+1}(t,z)$$

estimate (2.18) follows from inequality (2.20).

As remarked earlier, inequality (2.18) also implies the estimate

$$\mu_{m+1}(z)<0,\quad \forall z\in K$$

Consider the identity

$$\mu_{m+1}(z)\left(s-\frac{M_m(z)}{-\mu_{m+1}(z)}\right)=s\overline{\lim_{t\to+\infty}}\frac{1}{t}\ln\omega_{m+1}(t,z)+(1-s)\overline{\lim_{t\to+\infty}}\frac{1}{t}\ln\omega_m(t,z) \tag{2.21}$$

We write estimate (2.19) with the substitution $m\to m-1$:

$$\omega_m(t,z)\leqslant C\prod_{j=m+1}^{n}\beta_j^{-1}\exp[(-\gamma_0+\Lambda_{m+1}^n)t],\quad \forall t\geqslant 0 \tag{2.22}$$

We deduce from identity (2.21) and from estimates (2.19) and (2.22) that

$$\mu_{m+1}(z)\left(s-\frac{M_m(z)}{-\mu_{m+1}(z)}\right)\leqslant s(-\gamma_0+\Lambda_{m+2}^n)+(1-s)(-\gamma_0+\Lambda_{m+1}^n)$$

Hence, from inequality (2.16) and the fact that $\mu_{m+1}(z)$ is negative, we obtain the estimate

$$s\geqslant\frac{M_m(z)}{|\mu_{m+1}(z)|},\quad \forall z\in K \tag{2.23}$$

In the case $\sup_K M_m(z)<0$, it follows directly from the definition of Lyapunov dimension that $\dim_L(K)\leqslant m$, which proves the theorem. In the case $\sup_K M_m(z)\geqslant 0$, estimate (2.17) follows from inequality (2.23).

The treatment of the case $m=n-1$ is similar, except that instead of inequality (2.19) we need only write the estimate

$$\omega_{m+1}(t,z)\leqslant C\exp(-\gamma_0 t)$$

Theorem 2. Let the pair (A,B) be completely controllable and the pairs (A,Q_k) completely observable. Suppose Conditions 1 and 2 of Lemma 5 hold for some sequence of numbers $\lambda_1\geqslant\ldots\geqslant\lambda_n$.

Let us assume that the following inequalities hold for some natural number $m\in[1,n-1]$ and some $s\in[0,1]$

$$\lambda_{n-m}>0,\quad \Lambda_{n-m+1}^n+s\lambda_{n-m}>0 \tag{2.24}$$

Then estimate (2.17) holds.

Proof. We will first show that inequality (2.18) holds. To do this we note that the fact that λ_{n-m} is positive and the truth of inequality (2.24) imply the estimate $\Lambda_{n-m}^n>0$. It follows from estimate (2.12) that

$$\omega_{m+1}(t,z)\leqslant\prod_{n-m}^{n}\beta_j\exp(-\Lambda_{n-m}^n t) \tag{2.25}$$

This at once implies inequality (2.18).

It is also clear that

$$\omega_m(t,z)\leqslant\prod_{n-m+1}^{n}\beta_j\exp(-\Lambda_{n-m+1}^n t) \tag{2.26}$$

Estimate (2.23) follows from identity (2.21) and inequalities (2.25) and (2.26). The rest of the proof of Theorem 2 is exactly the same as the corresponding reasoning employed in the proof of Theorem 1.

3. LOCALIZATION BOUNDS FOR GLOBAL ATTRACTORS OF THE LORENZ SYSTEM

Consider the Lorenz system

$$\dot{X} = -d(X - Y), \quad \dot{Y} = rX - Y - XZ, \quad \dot{Z} = -bZ + XY \tag{3.1}$$

where d, r and b are positive numbers. Suppose, in addition, that $r > 1$ and $2d > b$. Note that if one of these conditions fails to hold, system (3.1) will be globally asymptotically stable [11, 15], that is, any of its solutions will tend to some equilibrium state as $t \to +\infty$.

Together with system (3.1), we will consider the equivalent system

$$\dot{\sigma} = \eta, \quad \dot{\eta} = -\mu\eta - \xi\sigma - \varphi(\sigma), \quad \dot{\xi} = -\alpha\xi - \beta\sigma\eta \tag{3.2}$$

where

$$\varphi(\sigma) = -\sigma + \gamma\sigma^3, \quad \sigma = \frac{\varepsilon X}{\sqrt{2d}}, \quad \eta = \varepsilon^2 (Y - X)\sqrt{2}, \quad \xi = \varepsilon^2 \left(Z - \frac{X^2}{b} \right)$$

$$t = \frac{t_1 \sqrt{d}}{\varepsilon}, \quad \mu = \frac{\varepsilon(d+1)}{\sqrt{d}}, \quad \alpha = \frac{\varepsilon b}{\sqrt{d}}, \quad \varepsilon = (r-1)^{-1/2}$$

$$\beta = 2\left(\frac{2d}{b} - 1 \right), \quad \gamma = \frac{2d}{b}$$

It is well known [11, 12, 15] that the surfaces

$$\psi_1 = \{ (r - Z)^2 + Y^2 = M^2 + \rho \} \quad \text{and} \quad \psi_2 = \{ Z - X^2/(2d) = -\rho \}$$

where $\rho > 0$ and

$$M = \begin{cases} r & \text{for} \quad b \leqslant 2 \\ br/(2\sqrt{b-1}) & \text{for} \quad b \geqslant 2 \end{cases}$$

are contact-free for solutions of system (3.1). Hence the following inequalities hold on a global attractor of system (3.1)

$$(r - Z)^2 + Y^2 \leqslant M^2 \tag{3.3}$$

$$Z \geqslant X^2/(2d) \tag{3.4}$$

Hence it follows that the following inequalities hold on a global attractor of system (3.2)

$$-\frac{M}{\sqrt{2}(r-1)} - \frac{\sqrt{d}\sigma}{\sqrt{r-1}} \leqslant \eta \leqslant \frac{M}{\sqrt{2}(r-1)} - \frac{\sqrt{d}\sigma}{\sqrt{r-1}} \tag{3.5}$$

$$\xi > -\beta\sigma^2/2 \quad \text{for} \quad \sigma \neq 0 \tag{3.6}$$

Using estimate (3.4), we introduce the comparison system [15, 16]

$$\dot{\sigma} = \eta, \quad \dot{\eta} = -\mu\eta + \sigma - \sigma^3 \tag{3.7}$$

which is equivalent to the first-order equation

$$P\frac{dP}{d\sigma}+\mu P-\sigma+\sigma^3=0 \tag{3.8}$$

The solutions $P_1(\sigma)$ of this equation with initial data $P_1(\sigma_0) = 0$, which are positive on the set $[0, \sigma_0]$, define the following contact-free surfaces of system (3.2) in the half-space $\{\sigma \geqslant 0\}$

$$\{\eta = P_1(\sigma),\ \ \eta > 0,\ \ \sigma \in [0, \sigma_0]\} \tag{3.9}$$

$$\{\eta < 0,\ \ \sigma = \sigma_0\} \tag{3.10}$$

The solutions $P_2(\sigma)$ of this equation with initial data $P_2(\sigma_0) = 0$, which are negative on the set $(-\sigma_0, 0)$, define the following contact-free surfaces of system (3.2) in the half-space $\{\sigma \leqslant 0\}$

$$\{\eta = P_2(\sigma),\ \ \eta < 0,\ \ \sigma \in [-\sigma_0, 0]\} \tag{3.11}$$

$$\{\eta > 0,\ \ \sigma = -\sigma_0\} \tag{3.12}$$

By estimate (3.5), it therefore follows that if the graph of $P = P_1(\sigma)$ cuts the graph of the straight line

$$P=\frac{M}{\sqrt{2}(r-1)}-\frac{\sqrt{d}\sigma}{\sqrt{r-1}}$$

at some point σ_1 of the interval $(0, \sigma_0)$, the following inequalities hold on a global attractor K of system (3.2)

$$\sigma < \sigma_0,\ \ \eta < P_1(\sigma)\ \ \text{for}\ \ \sigma \in [\sigma_1, \sigma_0] \tag{3.13}$$

Similarly, if the graph of $P = P_2(\sigma)$ cuts the graph of the straight line

$$P=-\frac{M}{\sqrt{2}(r-1)}-\frac{\sqrt{d}\sigma}{\sqrt{r-1}}$$

at some point σ_2 in the interval $(-\sigma_0, 0)$ the following inequalities hold on a global attractor K of system (3.2)

$$\sigma > -\sigma_0,\ \ \eta > P_2(\sigma)\ \ \text{for}\ \ \sigma \in [-\sigma_0,\ \sigma_2] \tag{3.14}$$

Note that in the strip $\{|\sigma| \leqslant \sigma_0\}$ the surfaces $\{\xi = C - \beta\sigma^2/2.\ C > \beta\sigma_0^2/2\}$ are contact-free for system (3.2). Hence the following estimate holds on a global attractor of system (3.2)

$$\xi \leqslant \beta(\sigma_0^2-\sigma^2)/2 \tag{3.15}$$

We have thus proved the following result.

Theorem 3. Estimates (3.3)–(3.6), (3.13)–(3.15) hold on a global attractor of system (3.1).

A similar result also holds for a global B-attractor of system (3.2).

We present one simple estimate of the number σ_0. To do this we note that, for inequalities (3.13) to hold, it is sufficient that the graphs $P = P_1(\sigma)$ should intersect the straight line $P = M/(\sqrt{2}(r-1))$.

Since the number μ in Eq. (3.8) is positive, we have

$$P_1(\sigma)^2 > (\sigma^2-\sigma_0^2)-\frac{1}{2}(\sigma^4-\sigma_0^4)$$

Therefore, a sufficient condition for the above intersection to take place is that

$$(1-\sigma_0^2)-\frac{1}{2}(1-\sigma_0^4)=\frac{M^2}{2(r-1)^2}$$

This inequality implies that

$$\sigma_0 = \sqrt{1 + \frac{M}{r-1}} \tag{3.16}$$

Similar reasoning may also be applied to estimate (3.14).

It follows from (3.16) that any global attractor of system (3.2) lies in a domain which is bounded uniformly with respect to the parameter $r \in (1, +\infty)$. For global B-attractors in the case $b \leqslant 2$, estimates (3.5), (3.13) and (3.14) are asymptotically the best possible as $r \to +\infty$. Indeed, in that case, as $r \to +\infty$ the following inequalities hold on a B-attractor

$$|\eta| \leqslant 1/\sqrt{2}, \quad |\sigma| \leqslant \sqrt{2}$$

We recall that part of a B-attractor consists of unstable manifolds of the zero equilibrium state, which may be represented in the zeroth approximation (for small ε) by the formulae

$$\{\xi = -\beta\sigma^2/2, \quad \eta^2 = \sigma^2 - \sigma^4/2\}$$

Hence for large r a B-attractor has points close to the planes $\{|\sigma| = \sqrt{2}\}$, $\{|\eta| = 1/\sqrt{2}\}$.

4. BOUNDS FOR THE LYAPUNOV DIMENSION OF AN ATTRACTOR OF THE LORENZ SYSTEM

We will now apply the Frequency Theorem (Theorem 1) and Localization Theorem (Theorem 3) to system (3.1).

We write system (3.1) in the form of (2.13), where

$$A = \begin{Vmatrix} -d & d & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -b \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad g(x) = \begin{Vmatrix} (r-Z)X \\ XY \end{Vmatrix}$$

We construct an Hermitian form as follows:

$$F_k(z, \xi) = \bar{\xi}_1 z_2 + \bar{\xi}_2 z_3 + \xi_1 \bar{z}_2 + \xi_2 \bar{z}_3 +$$
$$+ M(\varkappa |z_1|^2 + \varkappa^{-1}|z_2|^2 + \varkappa^{-1}|z_3|^2)$$

where ξ_j and z_j are the components of the vectors ξ and z, and M and $\varkappa$ are certain positive parameters to be determined later.

In the case under consideration, inequality (2.15) takes the form

$$2[(r-Z)y_2 + Yy_3]y_1 + M(\varkappa|y_1|^2 + \varkappa^{-1}|y_2|^2 + \varkappa^{-1}|y_3|^2) \geqslant 0$$
$$\forall y_j \in \mathbf{R}^1, \quad j = 1, 2, 3$$

This inequality will hold if

$$(r-Z)^2 + Y^2 \leqslant M^2$$

This inequality is identical with estimate (3.3).

In the case under consideration,

$$[pI - A]^{-1}B = \begin{Vmatrix} \dfrac{d}{(p+d)(p+1)}\xi_1 \\ \dfrac{1}{p+1}\xi_1 \\ \dfrac{1}{p+b}\xi_2 \end{Vmatrix}, \quad p \in C^1$$

Consequently, condition (2.8) becomes

$$2(\lambda_k - 1) \geqslant \varkappa^{-1} M + \varkappa d^2 M / (\lambda_k - d)^2$$

$$2(\lambda_k - b) \geqslant \varkappa^{-1} M$$

Taking $k = 3$, $\varkappa = d^{-1}(\lambda_3 - d)$, we write inequality (2.16) in the form

$$\lambda_3 (1 - s) < d + b + 1$$

Hence, Theorem 1 implies the following.

Corollary 1. If $d \geqslant b - 2$ and for some $s \in [0, 1]$

$$\left(\frac{d+b+1}{1-s} - 1\right)\left(\frac{d+b+1}{1-s} - d\right) > Md$$

then $\dim_L K \leqslant 2 + s$.

This inequality was established previously [5], subject to the additional condition that $1 < b < d$.

5. THE EXISTENCE OF HOMOCLINIC ORBITS IN THE LORENZ SYSTEM

Let $\sigma(t)^+$, $\xi(t)^+$, $\eta(t)^+$ denote the separatrix of the saddle point $\sigma = \xi = \eta = 0$ that goes into the half-plane $\{\sigma > 0\}$, that is, a solution of system (3.2) such that

$$\lim_{t \to -\infty} \sigma(t)^+ = \lim_{t \to -\infty} \xi(t)^+ = \lim_{t \to -\infty} \eta(t)^+ = 0$$

and $\sigma(t)^+ > 0$ for $t \in (-\infty, \mathrm{T})$, where T is a real number or $+\infty$. It is well known [15–17] that if the parameters d and b are fixed and the parameter r is sufficiently close to unity, then $T = +\infty$.

Definition 7. If

$$\lim_{t \to +\infty} \sigma(t)^+ = \lim_{t \to +\infty} \xi(t)^+ = \lim_{t \to +\infty} \eta(t)^+ = 0$$

then the trajectory $\sigma(t)^+$, $\xi(t)^+$, $\eta(t)^+$ will be called a *homoclinic orbit*.

Let us consider a smooth path $b(s)$, $d(s)$, $r(s)$ ($s \in [0, 1]$) in the parameter space $\{b, d, r\}$.

The main result of this section is the following.

Theorem 4. Suppose for system (3.2) with parameters $b(0)$, $d(0)$, $r(0)$ numbers $T > \tau$ exist such that

$$\sigma(T)^+ = \eta(\tau)^+ = 0 \tag{5.1}$$

$$\sigma(t)^+ > 0, \quad \forall t < T \tag{5.2}$$

$$\eta(t)^+ \neq 0, \quad \forall t < T, \quad t \neq \tau \tag{5.3}$$

Assume in addition that for system (3.2) with parameters $b(1)$, $d(1)$ and $r(1)$

$$\sigma(t)^+ > 0, \quad \forall t \in (-\infty, +\infty) \tag{5.4}$$

Then a number $s_0 \in [0, 1]$ exists such that system (3.2) with parameters $b(s_0)$, $d(s_0)$ and $r(s_0)$ has a homoclinic orbit $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$.

For the proof of this proposition, we need the following lemmas.

Lemma 6. If the following conditions hold for system (3.2)

$$\eta(\tau)^+ = 0, \quad \eta(t)^+ > 0, \quad \forall t \in (-\infty, \tau)$$

then $\dot{\eta}(\tau)^+ < 0$.

Proof. Suppose the contrary, i.e. $\dot{\eta}(\tau)^+ = 0$. Then it follows from the last two equations of system (3.2) that

$$\ddot{\eta}(\tau)^+ = \alpha\xi(\tau)^+\sigma(\tau)^+ \tag{5.5}$$

It follows from the relations $\eta(t)^+ > 0$, $\sigma(t)^+ > 0$, $\forall t \in (-\infty, \tau)$ and from the last equation of system (3.2) that $\xi(t)^+ < 0, \forall t \in (-\infty, \tau)$. This inequality and (5.5) imply the inequality $\ddot{\eta}(\tau)^+ < 0$. But this contradicts the assumption $\dot{\eta}(\tau)^+ = 0$ and the conditions of the lemma, proving Lemma 6.

Lemma 7. Given system (3.2), suppose that relations (5.1) and (5.2) are true and moreover

$$\begin{aligned} &\eta(t)^+ > 0, \quad \forall t \in (-\infty, \tau) \\ &\eta(t)^+ \leqslant 0, \quad \forall t \in (\tau, T) \end{aligned} \tag{5.6}$$

Then inequality (5.3) also holds.

Proof. Supposing the contrary, we conclude that a number $\rho \in (\tau, T)$ exists such that

$$\begin{aligned} &\eta(\rho)^+ = \dot{\eta}(\rho)^+ = 0 \\ &\ddot{\eta}(\rho)^+ = \alpha\sigma(\rho)^+\xi(\rho)^+ < 0 \\ &\eta(t)^+ < 0, \quad \forall t \in (\rho, T) \end{aligned}$$

Hence, from conditions (5.1) and (5.2) and the fact that the trajectory $\sigma(t) = \eta(t) = 0$, $\xi(t) = \xi(0)\exp(-\alpha t)$ belongs to the stable manifold of the saddle point $\sigma = \eta = \xi = 0$, we infer that the separatrix $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$ intersects this stable manifold. But then the separatrix must be a subset of the stable manifold of the saddle point. At the same time, we have $\sigma(t)^+ > 0$, $\forall t \geqslant \rho$. This contradicts condition (5.1), proving Lemma 7.

The proof of Lemma 7 admits of the following geometric interpretation in the phase space with coordinates σ, η, ξ. Situated "beneath" the set $\{\sigma > 0, \eta = 0, \xi \leqslant 1 - \gamma\sigma^2\}$ is a piece of the two-dimensional stable manifold of the saddle point $\sigma = \eta = \xi = 0$. This prevents trajectories with initial data in that set from reaching the plane $\{\sigma = 0\}$ while still in the quadrant $\{\sigma \geqslant 0, \eta \leqslant 0\}$.

Consider a polynomial

$$p^3 + ap^2 + bp + c \tag{5.7}$$

where a, b and c are positive numbers.

Lemma 8. Either all zeros of the polynomial (5.7) have negative real parts, or two of them have zero imaginary parts.

Proof. It is well known [14] that all the zeros of the polynomial (5.7) have negative real parts if and only if $ab > c$. If $ab = c$, the polynomial has two pure imaginary zeros.

Now let us assume that for some a, b and c with $ab < c$ polynomial (5.7) has only real zeros. Since the coefficients are positive, it follows that these zeros are negative. This leads to the inequality $ab > c$, which contradicts the above statement.

Proof of Theorem 4. It is known [18] that the semi-trajectory $\{\sigma(t)^+, \eta(t)^+, \xi(t)^+ \mid t \in (-\infty, t_0)\}$ depends continuously on the parameter s (t_0 is an arbitrary fixed number). If follows from this and from Lemma 6 that, if conditions (5.1)–(5.3) hold for system (3.2) with parameters $b(s_1)$, $d(s_1)$ and $r(s_1)$, then they also hold for $b(s)$, $d(s)$ and $r(s)$, provided that $s \in (s_1 - \delta, s_1 + \delta)$, where δ is some sufficiently small number, and the numbers τ and T depend on the parameter s.

It follows from the above arguments that relations (5.1)–(5.3) hold in some interval $(0, s_0)$. Henceforth we will assume that $(0, s_0)$ is the maximum interval in which these relations hold.

We claim that the values of the parameters $b(s_0)$, $d(s_0)$, $r(s_0)$ determine a homoclinic orbit.

We first note that for these parameters and some value of τ

$$\begin{gathered}\eta(t)^+ > 0, \quad \forall t < \tau, \quad \eta(t)^+ \leqslant 0, \quad \forall t \geqslant \tau \\ \sigma(t)^+ > 0, \quad \forall t \in (-\infty, +\infty)\end{gathered} \tag{5.8}$$

Indeed, if numbers $T_2 > T_1 > \tau$ exist for which

$$\sigma(t)^+ > 0, \quad \forall t \in (-\infty, T_2); \quad \sigma(T_2)^+ = 0, \quad \eta(T_1)^+ > 0$$

$$\eta(t)^+ > 0, \quad \forall t < \tau; \quad \eta(\tau)^+ = 0, \quad \dot{\eta}(\tau)^+ < 0$$

then for values $s < s_0$ sufficiently close to s_0, the inequality $\eta(T_1)^+ > 0$ still holds. This contradicts the definition of the number s_0.

If numbers $T_1 > \tau$ exist such that

$$\eta(T_1)^+ > 0, \quad \eta(t)^+ > 0, \quad \forall t < \tau$$

$$\eta(\tau)^+ = 0, \quad \dot{\eta}(\tau)^+ < 0, \quad \sigma(t)^+ > 0, \quad \forall t \in (-\infty, +\infty)$$

then again, for $s < s_0$ sufficiently close to s_0, the inequality $\eta(T_1)^+ > 0$ still holds, which contradicts the definition of the number s_0.

If numbers $T > \tau$ exist such that

$$\sigma(t)^+ > 0, \quad \forall t < T, \quad \sigma(T)^+ = 0, \quad \eta(t)^+ > 0, \quad \forall t < \tau$$

$$\eta(t)^+ \leqslant 0, \quad \forall t \in [\tau, T]$$

then, by Lemma 7, inequality (5.3) holds. Consequently, relations (5.1)–(5.3) hold for $s = s_0$, and $(0, s_0)$ is not the maximum interval in which they hold.

These contradictions complete the proof of inequalities (5.8).

It follows from (5.8) that the ω-limit set of the trajectory $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$ at $s = s_0$ is necessarily an equilibrium state.

We will show that the equilibrium state $\sigma = 1\sqrt{\gamma}$, $\eta = \xi = 0$ cannot be an ω-limit point of the trajectory in question.

Linearizing in the neighbourhood of this equilibrium state, we obtain the following characteristic polynomial

$$p^3 + (\alpha + \mu)p^2 + (\alpha\mu + 2/\gamma)p + 2\alpha$$

Let us assume that when $s = s_0$ the ω-limit set of the separatrix $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$ contains the point $\sigma = 1\sqrt{\gamma}$, $\eta = \xi = 0$. Using Lemma 8 and the fact that the semi-trajectory $\{\sigma(t)^+, \eta(t)^+, \xi(t)^+ \mid t \in (-\infty, t_0)\}$ is a continuous function of the parameter s, we conclude that for s close to s_0 the separatrices $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$ either tend to the equilibrium state $\sigma = 1\sqrt{\gamma}$, $\eta = \xi = 0$ at $t \to +\infty$, or oscillate in some time interval with changing sign of the coordinate η. Both these possibilities contradict properties (5.1)–(5.3).

Hence, for system (3.2) with parameters $b(s_0)$, $d(s_0)$, $r(s_0)$, the trajectory $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$ tends to the zero equilibrium state as $t \to +\infty$.

Note that the proof of Theorem 2 actually yields a stronger result, which may be formulated as follows.

If relations (5.1)–(5.3) hold for $s \in [0, s_0]$, but not for $s = s_0$, then system (1.2) with parameters $b(s_0)$, $d(s_0)$, $r(s_0)$ has a homoclinic orbit.

Let us apply Theorem 4 in various specific cases.

Fix the numbers b and d. It is well known [15–17] that inequality (5.4) is true for values of r sufficiently close to unity. We will show that if

$$3d - 2b > 1 \tag{5.9}$$

and r is sufficiently large, then relations (5.1)–(5.3) will hold. Indeed, consider the system

$$Q\frac{dQ}{d\sigma} = -\mu Q - P\sigma - \varphi(\sigma)$$
$$Q\frac{dP}{d\sigma} = -\alpha P - \beta Q\sigma \qquad (5.10)$$

which is equivalent to system (3.2) in the sets $\{\sigma \geqslant 0, \eta > 0\}$ and $\{\sigma \geqslant 0, \eta < 0\}$, where P and Q are solutions of system (5.10) which are functions of σ.

Since Theorem 3 implies that the quantities $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$ are bounded uniformly with respect to the parameter r, we can carry out an asymptotic integration of the solutions of system (5.10) with a small parameter ε that correspond to the separatrix under consideration. In the first approximation, these solutions may be written in the form

$$Q_1^2(\sigma) = \sigma^2 - \frac{\sigma^4}{2} - 2\mu\int_0^\sigma \sigma R_\sigma d\sigma - 2\alpha\beta\int_0^\sigma \sigma(1 - R_\sigma)d\sigma$$

$$Q_1(\sigma) \geqslant 0, \quad P_1(\sigma) = -\left(\frac{\beta}{2}\right)\sigma^2 + \alpha\beta(1 - R_\sigma)$$

$$Q_2^2(\sigma) = \sigma^2 - \frac{\sigma^4}{2} - 2\mu\int_\sigma^{\sqrt{2}} \sigma R_\sigma d\sigma - \frac{4}{3}\mu + 2\alpha\beta\int_\sigma^{\sqrt{2}} \sigma(1 + R_\sigma)d\sigma - \frac{2}{3}\alpha\beta$$

$$Q_2(\sigma) \leqslant 0, \quad P_2(\sigma) = -\left(\frac{\beta}{2}\right)\sigma^2 + \alpha\beta(1 + R_\sigma); \quad R_\sigma = \sqrt{1 - \frac{\sigma^2}{2}}$$

It follows from these formulae that, if inequality (5.9) holds, then for some $T > \tau$ relations (5.1)–(5.3) will also hold, and at the same time

$$\xi(T)^+ = P_2(0) = 2\alpha\beta$$
$$\eta(T)^+ = Q_2(0) = -\sqrt{8(\alpha\beta - \mu)/3} = -\sqrt{8\varepsilon(3d - 2b - 1)/(3\sqrt{d})}$$

Thus, all the conditions of Theorem 4 hold for the special path $b(s) \equiv b$, $d(s) \equiv d$, $r(s) : r(0) = r_1$, $r(1) = r_2$, where r_1 is sufficiently large and r_2 is fairly close to unity. We may therefore formulate the following result.

Corollary 2. For any positive numbers b and d satisfying inequality (5.9), a number $r \in (1, +\infty)$ exists such that system (3.2) with these parameters b, d and r has a homoclinic orbit $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$.

This result was first obtained in [19] and discussed later in [20–23].

Now fix $d = 10$ and $r = 28$, and consider the parameter $b \in (0, +\infty)$. It is well known [23] that when

$$b > (3d - 1)/2$$

condition (5.4) is satisfied. To analyse system (3.2) for small b, we reduce it to the form

$$\dot\sigma = \eta$$
$$\dot\eta = -\mu\eta - u\sigma + \sigma - \sigma^3 \qquad (5.11)$$
$$\dot u = -\alpha u + \varepsilon[(2d - b)/\sqrt{d}]\sigma^2$$

where $u = \xi + \beta\sigma^2/2$.

Since the semi-trajectory $\{\sigma(t)^+, \eta(t)^+, \xi(t)^+ \mid t \in (-\infty, t_0)\}$ depends continuously on the parameter b, it follows that, when b is small, system (5.11) may be replaced by the following "limiting" equations

$$\dot\sigma = \eta$$
$$\dot\eta = -\varepsilon[(d + 1)/\sqrt{d}]\eta - u\sigma + \sigma - \sigma^3 \qquad (5.12)$$
$$\dot u = 2\varepsilon\sqrt{d}\sigma^2$$

Numerical integration of the solution $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$ of system (5.12) for $d = 10, r = 28$ shows that conditions (5.1)–(5.3) are satisfied.

Hence, the above arguments, using Theorem 4, yield the following

Corollary 3. Let $d = 10$ and $r = 28$. Positive number b_0 exists such that system (3.2) with parameters $b = b_0$, $d = 10$ and $r = 28$ has a homoclinic orbit $\sigma(t)^+$, $\eta(t)^+$, $\xi(t)^+$.

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